CHAPTER 2
Logarithmic and Exponential Functions

EXERCISE SET 2.1

1. **(b)** $A = x^2$ **(c)** $\dfrac{dA}{dt} = 2x\dfrac{dx}{dt}$

(d) Find $\left.\dfrac{dA}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = 2$. From part (c), $\left.\dfrac{dA}{dt}\right|_{x=3} = 2(3)(2) = 12 \text{ ft}^2/\text{min}$.

2. **(b)** $A = \pi r^2$ **(c)** $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$

(d) Find $\left.\dfrac{dA}{dt}\right|_{r=5}$ given that $\left.\dfrac{dr}{dt}\right|_{r=5} = 2$. From part (c), $\left.\dfrac{dA}{dt}\right|_{r=5} = 2\pi(5)(2) = 20\pi \text{ cm}^2/\text{s}$.

3. **(a)** $V = \pi r^2 h$, so $\dfrac{dV}{dt} = \pi\left(r^2\dfrac{dh}{dt} + 2rh\dfrac{dr}{dt}\right)$.

(b) Find $\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}}$ given that $\left.\dfrac{dh}{dt}\right|_{\substack{h=6,\\ r=10}} = 1$ and $\left.\dfrac{dr}{dt}\right|_{\substack{h=6,\\ r=10}} = -1$. From part (a),

$\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}} = \pi[10^2(1) + 2(10)(6)(-1)] = -20\pi \text{ in}^3/\text{s}$; the volume is decreasing.

4. **(a)** $\ell^2 = x^2 + y^2$, so $\dfrac{d\ell}{dt} = \dfrac{1}{\ell}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$.

(b) Find $\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\ y=4}}$ given that $\dfrac{dx}{dt} = \dfrac{1}{2}$ and $\dfrac{dy}{dt} = -\dfrac{1}{4}$.

From part (a) and the fact that $\ell = 5$ when $x = 3$ and $y = 4$,

$\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\ y=4}} = \dfrac{1}{5}\left[3\left(\dfrac{1}{2}\right) + 4\left(-\dfrac{1}{4}\right)\right] = \dfrac{1}{10}$ ft/s; the diagonal is increasing.

5. **(a)** $\tan\theta = \dfrac{y}{x}$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$, $\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{x^2}\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right)$

(b) Find $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\ y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=2,\\ y=2}} = 1$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\ y=2}} = -\dfrac{1}{4}$.

When $x = 2$ and $y = 2$, $\tan\theta = 2/2 = 1$ so $\theta = \dfrac{\pi}{4}$ and $\cos\theta = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$. Thus from part (a), $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\ y=2}} = \dfrac{(1/\sqrt{2})^2}{2^2}\left[2\left(-\dfrac{1}{4}\right) - 2(1)\right] = -\dfrac{5}{16}$ rad/s; θ is decreasing.

6. Find $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\ y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=1,\\ y=2}} = -2$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\ y=2}} = 3$.

$\dfrac{dz}{dt} = 2x^3y\dfrac{dy}{dt} + 3x^2y^2\dfrac{dx}{dt}$, $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\ y=2}} = (4)(3) + (12)(-2) = -12$ units/s; z is decreasing

7. Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\dfrac{dA}{dt}$ given that $\dfrac{d\theta}{dt} = \dfrac{\pi}{30}$ rad/min; $A = \dfrac{1}{2}r^2\theta = 8\theta$, so $\dfrac{dA}{dt} = 8\dfrac{d\theta}{dt} = \dfrac{4\pi}{15}$ in^2/min.

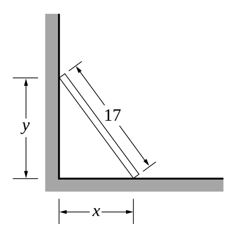
8. Let r be the radius and A the area enclosed by the ripple. We want $\left.\dfrac{dA}{dt}\right|_{t=10}$ given that $\dfrac{dr}{dt} = 3$. We know that $A = \pi r^2$, so $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. Because r is increasing at the constant rate of 3 ft/s, it follows that $r = 30$ ft after 10 seconds so $\left.\dfrac{dA}{dt}\right|_{t=10} = 2\pi(30)(3) = 180\pi$ ft^2/s.

9. Find $\left.\dfrac{dr}{dt}\right|_{A=9}$ given that $\dfrac{dA}{dt} = 6$. From $A = \pi r^2$ we get $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$ so $\dfrac{dr}{dt} = \dfrac{1}{2\pi r}\dfrac{dA}{dt}$. If $A = 9$ then $\pi r^2 = 9$, $r = 3/\sqrt{\pi}$ so $\left.\dfrac{dr}{dt}\right|_{A=9} = \dfrac{1}{2\pi(3/\sqrt{\pi})}(6) = 1/\sqrt{\pi}$ mi/h.

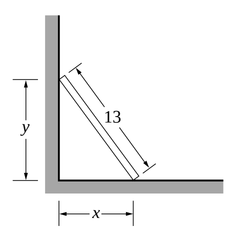
10. The volume V of a sphere of radius r is given by $V = \dfrac{4}{3}\pi r^3$ or, because $r = \dfrac{D}{2}$ where D is the diameter, $V = \dfrac{4}{3}\pi\left(\dfrac{D}{2}\right)^3 = \dfrac{1}{6}\pi D^3$. We want $\left.\dfrac{dD}{dt}\right|_{r=1}$ given that $\dfrac{dV}{dt} = 3$. From $V = \dfrac{1}{6}\pi D^3$ we get $\dfrac{dV}{dt} = \dfrac{1}{2}\pi D^2\dfrac{dD}{dt}$, $\dfrac{dD}{dt} = \dfrac{2}{\pi D^2}\dfrac{dV}{dt}$, so $\left.\dfrac{dD}{dt}\right|_{r=1} = \dfrac{2}{\pi(2)^2}(3) = \dfrac{3}{2\pi}$ ft/min.

11. Find $\left.\dfrac{dV}{dt}\right|_{r=9}$ given that $\dfrac{dr}{dt} = -15$. From $V = \dfrac{4}{3}\pi r^3$ we get $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$ so $\left.\dfrac{dV}{dt}\right|_{r=9} = 4\pi(9)^2(-15) = -4860\pi$. Air must be removed at the rate of 4860π cm^3/min.

12. Let x and y be the distances shown in the diagram. We want to find $\left.\dfrac{dy}{dt}\right|_{y=8}$ given that $\dfrac{dx}{dt} = 5$. From $x^2 + y^2 = 17^2$ we get $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$, so $\dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$. When $y = 8$, $x^2 + 8^2 = 17^2$, $x^2 = 289 - 64 = 225$, $x = 15$ so $\left.\dfrac{dy}{dt}\right|_{y=8} = -\dfrac{15}{8}(5) = -\dfrac{75}{8}$ ft/s; the top of the ladder is moving down the wall at a rate of 75/8 ft/s.

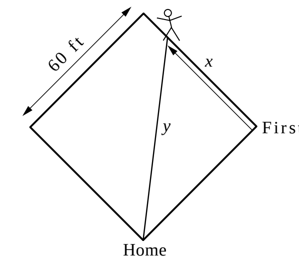


13. Find $\left.\dfrac{dx}{dt}\right|_{y=5}$ given that $\dfrac{dy}{dt} = -2$. From $x^2 + y^2 = 13^2$ we get $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ so $\dfrac{dx}{dt} = -\dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2 + y^2 = 169$ to find that $x = 12$ when $y = 5$ so $\left.\dfrac{dx}{dt}\right|_{y=5} = -\dfrac{5}{12}(-2) = \dfrac{5}{6}$ ft/s.

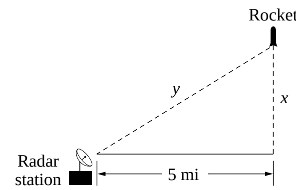


14. Let θ be the acute angle, and x the distance of the bottom of the plank from the wall. Find $\left.\dfrac{d\theta}{dt}\right|_{x=2}$ given that $\left.\dfrac{dx}{dt}\right|_{x=2} = -\dfrac{1}{2}$ ft/s. The variables θ and x are related by the equation $\cos\theta = \dfrac{x}{10}$ so $-\sin\theta\dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dx}{dt}$, $\dfrac{d\theta}{dt} = -\dfrac{1}{10\sin\theta}\dfrac{dx}{dt}$. When $x = 2$, the top of the plank is $\sqrt{10^2 - 2^2} = \sqrt{96}$ ft above the ground so $\sin\theta = \sqrt{96}/10$ and $\left.\dfrac{d\theta}{dt}\right|_{x=2} = -\dfrac{1}{\sqrt{96}}\left(-\dfrac{1}{2}\right) = \dfrac{1}{2\sqrt{96}} \approx 0.051$ rad/s.

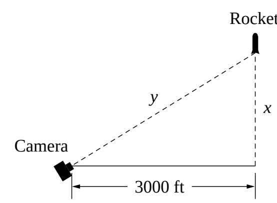
15. Let x denote the distance from first base and y the distance from home plate. Then $x^2 + 60^2 = y^2$ and $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$. When $x = 50$ then $y = 10\sqrt{61}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{50}{10\sqrt{61}}(25) = \dfrac{125}{\sqrt{61}}$ ft/s.



16. Find $\left.\dfrac{dx}{dt}\right|_{x=4}$ given that $\left.\dfrac{dy}{dt}\right|_{x=4} = 2000$. From $x^2 + 5^2 = y^2$ we get $2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$ so $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt}$. Use $x^2 + 25 = y^2$ to find that $y = \sqrt{41}$ when $x = 4$ so $\left.\dfrac{dx}{dt}\right|_{x=4} = \dfrac{\sqrt{41}}{4}(2000) = 500\sqrt{41}$ mi/h.



17. Find $\left.\dfrac{dy}{dt}\right|_{x=4000}$ given that $\left.\dfrac{dx}{dt}\right|_{x=4000} = 880$. From $y^2 = x^2 + 3000^2$ we get $2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$ so $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt}$. If $x = 4000$, then $y = 5000$ so $\left.\dfrac{dy}{dt}\right|_{x=4000} = \dfrac{4000}{5000}(880) = 704$ ft/s.



18. Find $\left.\dfrac{dx}{dt}\right|_{\phi=\pi/4}$ given that $\left.\dfrac{d\phi}{dt}\right|_{\phi=\pi/4} = 0.2$. But $x = 3000\tan\phi$ so $\dfrac{dx}{dt} = 3000(\sec^2\phi)\dfrac{d\phi}{dt}$, $\left.\dfrac{dx}{dt}\right|_{\phi=\pi/4} = 3000\left(\sec^2\dfrac{\pi}{4}\right)(0.2) = 1200$ ft/s.

19. **(a)** If x denotes the altitude, then $r - x = 3960$, the radius of the Earth. $\theta = 0$ at perigee, so $r = 4995/1.12 \approx 4460$; the altitude is $x = 4460 - 3960 = 500$ miles. $\theta = \pi$ at apogee, so $r = 4995/0.88 \approx 5676$; the altitude is $x = 5676 - 3960 = 1716$ miles.

(b) If $\theta = 120^\circ$, then $r = 4995/0.94 \approx 5314$; the altitude is $5314 - 3960 = 1354$ miles. The rate of change of the altitude is given by

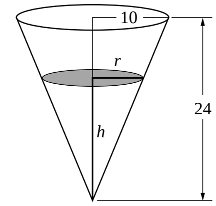
$$\frac{dx}{dt} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{4995(0.12\sin\theta)}{(1+0.12\cos\theta)^2}\frac{d\theta}{dt}.$$

Use $\theta = 120^\circ$ and $d\theta/dt = 2.7^\circ/\text{min} = (2.7)(\pi/180)$ rad/min to get $dr/dt \approx 27.7$ mi/min.

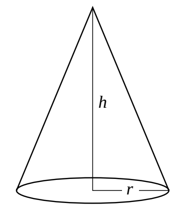
20. **(a)** Let x be the horizontal distance shown in the figure. Then $x = 4000\cot\theta$ and $\dfrac{dx}{dt} = -4000\csc^2\theta\dfrac{d\theta}{dt}$, so $\dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{4000}\dfrac{dx}{dt}$. Use $\theta = 30^\circ$ and $dx/dt = 300$ mi/h $= 300(5280/3600)$ ft/s $= 440$ ft/s to get $d\theta/dt = -0.0275$ rad/s $\approx -1.6^\circ$/s; θ is decreasing at the rate of 1.6°/s.

(b) Let y be the distance between the observation point and the aircraft. Then $y = 4000\csc\theta$ so $dy/dt = -4000(\csc\theta\cot\theta)(d\theta/dt)$. Use $\theta = 30^\circ$ and $d\theta/dt = -0.0275$ rad/s to get $dy/dt \approx 381$ ft/s.

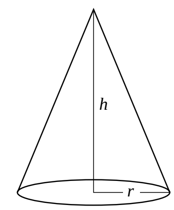
21. Find $\left.\dfrac{dh}{dt}\right|_{h=16}$ given that $\dfrac{dV}{dt} = 20$. The volume of water in the tank at a depth h is $V = \dfrac{1}{3}\pi r^2 h$. Use similar triangles (see figure) to get $\dfrac{r}{h} = \dfrac{10}{24}$ so $r = \dfrac{5}{12}h$ thus $V = \dfrac{1}{3}\pi\left(\dfrac{5}{12}h\right)^2 h = \dfrac{25}{432}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{25}{144}\pi h^2 \dfrac{dh}{dt}$; $\dfrac{dh}{dt} = \dfrac{144}{25\pi h^2}\dfrac{dV}{dt}$, $\left.\dfrac{dh}{dt}\right|_{h=16} = \dfrac{9}{20\pi}$ ft/min.



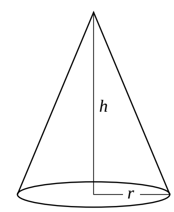
22. Find $\left.\dfrac{dh}{dt}\right|_{h=6}$ given that $\dfrac{dV}{dt} = 8$. $V = \dfrac{1}{3}\pi r^2 h$, but $r = \dfrac{1}{2}h$ so $V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$, $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt}$, $\dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt}$,
$\left.\dfrac{dh}{dt}\right|_{h=6} = \dfrac{4}{\pi(6)^2}(8) = \dfrac{8}{9\pi}$ ft/min.



23. Find $\left.\dfrac{dV}{dt}\right|_{h=10}$ given that $\dfrac{dh}{dt} = 5$. $V = \dfrac{1}{3}\pi r^2 h$, but $r = \dfrac{1}{2}h$ so
$V = \dfrac{1}{3}\pi\left(\dfrac{h}{2}\right)^2 h = \dfrac{1}{12}\pi h^3$,
$\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt}$, $\left.\dfrac{dV}{dt}\right|_{h=10} = \dfrac{1}{4}\pi(10)^2(5) = 125\pi$ ft^3/min.



24. Let r and h be as shown in the figure. If C is the circumference of the base, then we want to find $\left.\dfrac{dC}{dt}\right|_{h=8}$ given that $\dfrac{dV}{dt} = 10$. It is given that $r = \dfrac{1}{2}h$, thus $C = 2\pi r = \pi h$ so



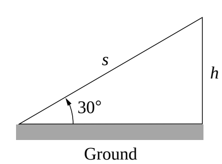
$$\frac{dC}{dt} = \pi\frac{dh}{dt} \qquad (1)$$

Use $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{12}\pi h^3$ to get $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt}$, so

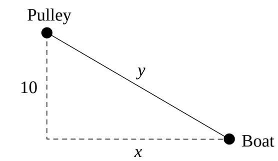
$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \qquad (2)$$

Substitution of (2) into (1) gives $\dfrac{dC}{dt} = \dfrac{4}{h^2}\dfrac{dV}{dt}$ so
$\left.\dfrac{dC}{dt}\right|_{h=8} = \dfrac{4}{64}(10) = \dfrac{5}{8}$ ft/min.

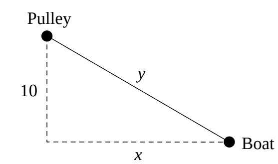
25. With s and h as shown in the figure, we want to find $\dfrac{dh}{dt}$ given that $\dfrac{ds}{dt} = 500$. From the figure, $h = s\sin 30^\circ = \dfrac{1}{2}s$ so
$\dfrac{dh}{dt} = \dfrac{1}{2}\dfrac{ds}{dt} = \dfrac{1}{2}(500) = 250$ mi/h.



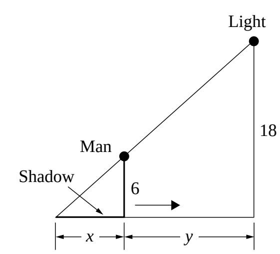
26. Find $\left.\frac{dx}{dt}\right|_{y=125}$ given that $\frac{dy}{dt} = -20$. From $x^2 + 10^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\left.\frac{dx}{dt}\right|_{y=125} = \frac{125}{15\sqrt{69}}(-20) = -\frac{500}{3\sqrt{69}}$. The boat is approaching the dock at the rate of $\frac{500}{3\sqrt{69}}$ ft/min.



27. Find $\frac{dy}{dt}$ given that $\left.\frac{dx}{dt}\right|_{y=125} = -12$. From $x^2 + 10^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$. Use $x^2 + 100 = y^2$ to find that $x = \sqrt{15,525} = 15\sqrt{69}$ when $y = 125$ so $\frac{dy}{dt} = \frac{15\sqrt{69}}{125}(-12) = -\frac{36\sqrt{69}}{25}$. The rope must be pulled at the rate of $\frac{36\sqrt{69}}{25}$ ft/min.

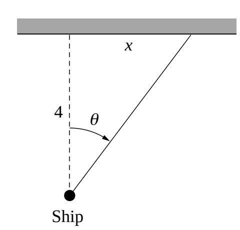


28. **(a)** Let x and y be as shown in the figure. It is required to find $\frac{dx}{dt}$, given that $\frac{dy}{dt} = -3$. By similar triangles, $\frac{x}{6} = \frac{x+y}{18}$, $18x = 6x + 6y$, $12x = 6y$, $x = \frac{1}{2}y$, so $\frac{dx}{dt} = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}(-3) = -\frac{3}{2}$ ft/s.

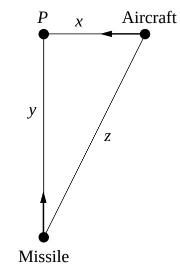


(b) The tip of the shadow is $z = x + y$ feet from the street light, thus the rate at which it is moving is given by $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$. In part (a) we found that $\frac{dx}{dt} = -\frac{3}{2}$ when $\frac{dy}{dt} = -3$ so $\frac{dz}{dt} = (-3/2) + (-3) = -9/2$ ft/s; the tip of the shadow is moving at the rate of $9/2$ ft/s toward the street light.

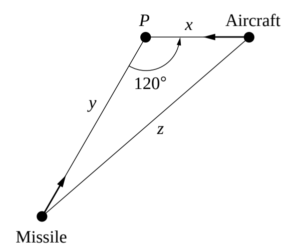
29. Find $\left.\frac{dx}{dt}\right|_{\theta=\pi/4}$ given that $\frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$ rad/s. Then $x = 4\tan\theta$ (see figure) so $\frac{dx}{dt} = 4\sec^2\theta\frac{d\theta}{dt}$, $\left.\frac{dx}{dt}\right|_{\theta=\pi/4} = 4\left(\sec^2\frac{\pi}{4}\right)\left(\frac{\pi}{5}\right) = 8\pi/5$ km/s.



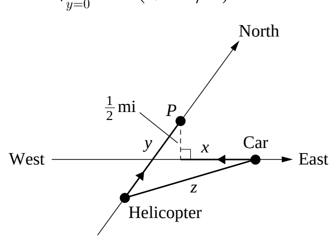
30. If x, y, and z are as shown in the figure, then we want $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=4}}$ given that $\dfrac{dx}{dt} = -600$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\ y=4}} = -1200$. But $z^2 = x^2 + y^2$ so $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$, $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$. When $x = 2$ and $y = 4$, $z^2 = 2^2 + 4^2 = 20$, $z = \sqrt{20} = 2\sqrt{5}$ so $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=4}} = \dfrac{1}{2\sqrt{5}}[2(-600) + 4(-1200)] = -\dfrac{3000}{\sqrt{5}} = -600\sqrt{5}$ mi/h; the distance between missile and aircraft is decreasing at the rate of $600\sqrt{5}$ mi/h.

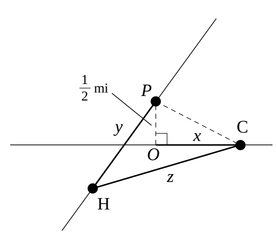


31. We wish to find $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=4}}$ given $\dfrac{dx}{dt} = -600$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\ y=4}} = -1200$ (see figure). From the law of cosines,
$z^2 = x^2 + y^2 - 2xy\cos 120° = x^2 + y^2 - 2xy(-1/2) = x^2 + y^2 + xy$, so
$2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + x\dfrac{dy}{dt} + y\dfrac{dx}{dt}$,
$\dfrac{dz}{dt} = \dfrac{1}{2z}\left[(2x + y)\dfrac{dx}{dt} + (2y + x)\dfrac{dy}{dt}\right]$. When $x = 2$ and $y = 4$,
$z^2 = 2^2 + 4^2 + (2)(4) = 28$, so $z = \sqrt{28} = 2\sqrt{7}$, thus
$\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=4}} = \dfrac{1}{2(2\sqrt{7})}[(2(2) + 4)(-600) + (2(4) + 2)(-1200)] = -\dfrac{4200}{\sqrt{7}} =$
$-600\sqrt{7}$ mi/h; the distance between missile and aircraft is decreasing at the rate of $600\sqrt{7}$ mi/h.



32. **(a)** Let x, y, and z be the distances shown in the first figure. Find $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=0}}$ given that $\dfrac{dx}{dt} = -75$ and $\dfrac{dy}{dt} = -100$. In order to find an equation relating x, y, and z, first draw the line segment that joins the point P to the car, as shown in the second figure. Because triangle OPC is a right triangle, it follows that PC has length $\sqrt{x^2 + (1/2)^2}$; but triangle HPC is also a right triangle so $z^2 = \left(\sqrt{x^2 + (1/2)^2}\right)^2 + y^2 = x^2 + y^2 + 1/4$ and $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} + 0$, $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$. Now, when $x = 2$ and $y = 0$, $z^2 = (2)^2 + (0)^2 + 1/4 = 17/4$, $z = \sqrt{17}/2$ so $\left.\dfrac{dz}{dt}\right|_{\substack{x=2,\\ y=0}} = \dfrac{1}{(\sqrt{17}/2)}[2(-75) + 0(-100)] = -300/\sqrt{17}$ mi/h





(b) decreasing, because $\dfrac{dz}{dt} < 0$.

33. **(a)** We want $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=1,\\y=2}} = 6$. For convenience, first rewrite the equation as $xy^3 = \dfrac{8}{5} + \dfrac{8}{5}y^2$ then $3xy^2\dfrac{dy}{dt} + y^3\dfrac{dx}{dt} = \dfrac{16}{5}y\dfrac{dy}{dt}$, $\dfrac{dy}{dt} = \dfrac{y^3}{\dfrac{16}{5}y - 3xy^2}\dfrac{dx}{dt}$ so $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}} = \dfrac{2^3}{\dfrac{16}{5}(2) - 3(1)2^2}(6) = -60/7$ units/s.

(b) falling, because $\dfrac{dy}{dt} < 0$

34. Find $\left.\dfrac{dx}{dt}\right|_{(2,5)}$ given that $\left.\dfrac{dy}{dt}\right|_{(2,5)} = 2$. Square and rearrange to get $x^3 = y^2 - 17$ so $3x^2\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$, $\dfrac{dx}{dt} = \dfrac{2y}{3x^2}\dfrac{dy}{dt}$, $\left.\dfrac{dx}{dt}\right|_{(2,5)} = \left(\dfrac{5}{6}\right)(2) = \dfrac{5}{3}$ units/s.

35. The coordinates of P are $(x, 2x)$, so the distance between P and the point $(3,0)$ is $D = \sqrt{(x-3)^2 + (2x-0)^2} = \sqrt{5x^2 - 6x + 9}$. Find $\left.\dfrac{dD}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = -2$. $\dfrac{dD}{dt} = \dfrac{5x-3}{\sqrt{5x^2-6x+9}}\dfrac{dx}{dt}$, so $\left.\dfrac{dD}{dt}\right|_{x=3} = \dfrac{12}{\sqrt{36}}(-2) = -4$ units/s.

36. **(a)** Let D be the distance between P and $(2,0)$. Find $\left.\dfrac{dD}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = 4$. $D = \sqrt{(x-2)^2 + y^2} = \sqrt{(x-2)^2 + x} = \sqrt{x^2 - 3x + 4}$ so $\dfrac{dD}{dt} = \dfrac{2x-3}{2\sqrt{x^2-3x+4}}$; $\left.\dfrac{dD}{dt}\right|_{x=3} = \dfrac{3}{2\sqrt{4}} = \dfrac{3}{4}$ units/s.

(b) Let θ be the angle of inclination. Find $\left.\dfrac{d\theta}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = 4$. $\tan\theta = \dfrac{y}{x-2} = \dfrac{\sqrt{x}}{x-2}$ so $\sec^2\theta\dfrac{d\theta}{dt} = -\dfrac{x+2}{2\sqrt{x}(x-2)^2}\dfrac{dx}{dt}$, $\dfrac{d\theta}{dt} = -\cos^2\theta\dfrac{x+2}{2\sqrt{x}(x-2)^2}\dfrac{dx}{dt}$. When $x = 3$, $D = 2$ so $\cos\theta = \dfrac{1}{2}$ and $\left.\dfrac{d\theta}{dt}\right|_{x=3} = -\dfrac{1}{4}\dfrac{5}{2\sqrt{3}}(4) = -\dfrac{5}{2\sqrt{3}}$ rad/s.

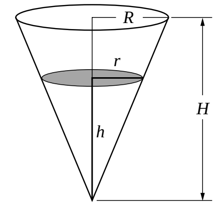
37. Solve $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ given $y = x\ln x$. Then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = (1+\ln x)\dfrac{dx}{dt}$, so $1 + \ln x = 3$, $\ln x = 2$, $x = e^2$.

38. $32x\dfrac{dx}{dt} + 18y\dfrac{dy}{dt} = 0$; if $\dfrac{dy}{dt} = \dfrac{dx}{dt} \neq 0$, then $(32x + 18y)\dfrac{dx}{dt} = 0$, $32x + 18y = 0$, $y = -\dfrac{16}{9}x$ so $16x^2 + 9\dfrac{256}{81}x^2 = 144$, $\dfrac{400}{9}x^2 = 144$, $x^2 = \dfrac{81}{25}$, $x = \pm\dfrac{9}{5}$. If $x = \dfrac{9}{5}$, then $y = -\dfrac{16}{9}\dfrac{9}{5} = -\dfrac{16}{5}$. Similarly, if $x = -\dfrac{9}{5}$, then $y = \dfrac{16}{5}$. The points are $(\dfrac{9}{5}, -\dfrac{16}{5})$ and $(-\dfrac{9}{5}, \dfrac{16}{5})$.

39. Find $\left.\dfrac{dS}{dt}\right|_{s=10}$ given that $\left.\dfrac{ds}{dt}\right|_{s=10} = -2$. From $\dfrac{1}{s} + \dfrac{1}{S} = \dfrac{1}{6}$ we get $-\dfrac{1}{s^2}\dfrac{ds}{dt} - \dfrac{1}{S^2}\dfrac{dS}{dt} = 0$, so $\dfrac{dS}{dt} = -\dfrac{S^2}{s^2}\dfrac{ds}{dt}$. If $s = 10$, then $\dfrac{1}{10} + \dfrac{1}{S} = \dfrac{1}{6}$ which gives $S = 15$. So $\left.\dfrac{dS}{dt}\right|_{s=10} = -\dfrac{225}{100}(-2) = 4.5$ cm/s. The image is moving away from the lens.

40. Suppose that the reservoir has height H and that the radius at the top is R. At any instant of time let h and r be the corresponding dimensions of the cone of water (see figure). We want to show that $\dfrac{dh}{dt}$ is constant and independent of H and R, given that $\dfrac{dV}{dt} = -kA$ where V is the volume of water, A is the area of a circle of radius r, and k is a positive constant. The volume of a cone of radius r and height h is $V = \dfrac{1}{3}\pi r^2 h$. By similar triangles $\dfrac{r}{h} = \dfrac{R}{H}$, $r = \dfrac{R}{H}h$ thus $V = \dfrac{1}{3}\pi\left(\dfrac{R}{H}\right)^2 h^3$ so

$$\frac{dV}{dt} = \pi\left(\frac{R}{H}\right)^2 h^2\frac{dh}{dt} \qquad (1)$$



But it is given that $\dfrac{dV}{dt} = -kA$ or, because $A = \pi r^2 = \pi\left(\dfrac{R}{H}\right)^2 h^2$, $\dfrac{dV}{dt} = -k\pi\left(\dfrac{R}{H}\right)^2 h^2$, which when substituted into equation (1) gives $-k\pi\left(\dfrac{R}{H}\right)^2 h^2 = \pi\left(\dfrac{R}{H}\right)^2 h^2\dfrac{dh}{dt}$, $\dfrac{dh}{dt} = -k$.

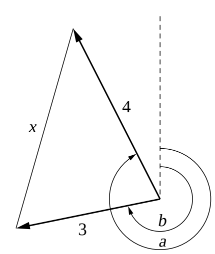
41. Let r be the radius, V the volume, and A the surface area of a sphere. Show that $\dfrac{dr}{dt}$ is a constant given that $\dfrac{dV}{dt} = -kA$, where k is a positive constant. Because $V = \dfrac{4}{3}\pi r^3$,

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \qquad (1)$$

But it is given that $\dfrac{dV}{dt} = -kA$ or, because $A = 4\pi r^2$, $\dfrac{dV}{dt} = -4\pi r^2 k$ which when substituted into equation (1) gives $-4\pi r^2 k = 4\pi r^2\dfrac{dr}{dt}$, $\dfrac{dr}{dt} = -k$.

42. Let x be the distance between the tips of the minute and hour hands, and α and β the angles shown in the figure. Because the minute hand makes one revolution in 60 minutes, $\dfrac{d\alpha}{dt} = \dfrac{2\pi}{60} = \pi/30$ rad/min; the hour hand makes one revolution in 12 hours (720 minutes), thus $\dfrac{d\beta}{dt} = \dfrac{2\pi}{720} = \pi/360$ rad/min. We want to find $\left.\dfrac{dx}{dt}\right|_{\substack{\alpha=2\pi,\\ \beta=3\pi/2}}$ given that $\dfrac{d\alpha}{dt} = \pi/30$ and $\dfrac{d\beta}{dt} = \pi/360$.

Using the law of cosines on the triangle shown in the figure,
$x^2 = 3^2 + 4^2 - 2(3)(4)\cos(\alpha - \beta) = 25 - 24\cos(\alpha - \beta)$, so
$2x\dfrac{dx}{dt} = 0 + 24\sin(\alpha - \beta)\left(\dfrac{d\alpha}{dt} - \dfrac{d\beta}{dt}\right)$,
$\dfrac{dx}{dt} = \dfrac{12}{x}\left(\dfrac{d\alpha}{dt} - \dfrac{d\beta}{dt}\right)\sin(\alpha - \beta)$. When $\alpha = 2\pi$ and $\beta = 3\pi/2$,
$x^2 = 25 - 24\cos(2\pi - 3\pi/2) = 25$, $x = 5$; so
$\left.\dfrac{dx}{dt}\right|_{\substack{\alpha=2\pi,\\ \beta=3\pi/2}} = \dfrac{12}{5}(\pi/30 - \pi/360)\sin(2\pi - 3\pi/2) = \dfrac{11\pi}{150}$ in/min.



43. Extend sides of cup to complete the cone and let V_0 be the volume of the portion added, then (see figure) $V = \dfrac{1}{3}\pi r^2 h - V_0$ where
$\dfrac{r}{h} = \dfrac{4}{12} = \dfrac{1}{3}$ so $r = \dfrac{1}{3}h$ and $V = \dfrac{1}{3}\pi\left(\dfrac{h}{3}\right)^2 h - V_0 = \dfrac{1}{27}\pi h^3 - V_0$,
$\dfrac{dV}{dt} = \dfrac{1}{9}\pi h^2\dfrac{dh}{dt}$, $\dfrac{dh}{dt} = \dfrac{9}{\pi h^2}\dfrac{dV}{dt}$,
$\left.\dfrac{dh}{dt}\right|_{h=9} = \dfrac{9}{\pi(9)^2}(20) = \dfrac{20}{9\pi}$ cm/s.